United States
Naval Postgraduate School

THESIS

REAL EIGENVALUES OF
UNSYMMETRIC MATRICES

by

Carl Sheldon Park, Jr.

Thesis Advisor: R. E. Ball

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Lieutenant, United States Navy
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ABSTRACT

This thesis is concerned with proving that the eigenvalues of a specific unsymmetric matrix are real and positive, without actually computing them. The method of finite differences is applied to the vibration analysis of a cantilever beam and leads to an unsymmetric stiffness matrix in the eigenvalue problem formulation. The technique employed in the proof is based on a perturbation theory given by Wilkinson for real symmetric matrices. Application of the theory is made to the cantilever beam eigenvalue problem. The results verify that the eigenvalues of this and other unsymmetric matrices can be proven real and positive without their actual values being calculated.

A multiple-variable formulation of the cantilever beam vibration analysis is also examined to illustrate that it implicitly contains all of the properties of a symmetric matrix.

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TABLE OF SYMBOLS

h	-	Subdivisions of beam, non-dimensional
i	-	Integer
j	-	Integer
m	-	Integer
n	-	Integer
r	-	Radius of gyration of beam, inches
s	-	Integer
t	-	Time, seconds
x	-	Horizontal coordinate, non-dimensional
A	-	Square symmetric matrix
B	-	Square unsymmetric matrix
E	-	Material modulus of elasticity, lbs/in^2
H	-	Modal matrix
I	-	Identity matrix
K	-	Beam stiffness matrix
L	-	Length of beam, inches
M	-	Beam mass matrix
$\mathcal{M}$	-	Beam bending moment
X	-	Horizontal coordinate, inches
Y	-	Vertical deflection, inches
Z	-	General square matrix
Φ	-	Vertical deflection as a function of x and t
λ	-	Eigenvalue, non-dimensional
μ	-	$=\rho/Er^2$

ρ - Mass density of beam, slugs/in.^{3}

ϕ - Natural mode

ω - Beam Natural Frequency, Rad/Sec

$[\ ]$ - Square Matrix

$\{\ \}$ - Column Matrix

$|\ |$ - Absolute Value

$\|\ \|_2$ - -2- Norm of Matrix (of page 15)

diag( ) - Diagonal Matrix

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I. INTRODUCTION

The vibration analysis of many structures can be accomplished by the application of finite differences to the partial differential equations governing the free undamped vibration of the structure. Assuming the motion of the structure to be harmonic, with natural frequency ω, leads to the matrix equation for the eigenvalue problem

$$K\phi = \omega^2 M \phi \qquad (1\text{-}1)$$

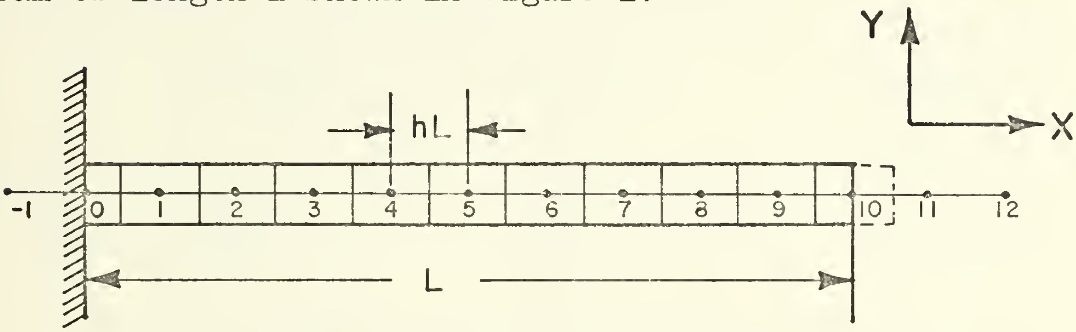
where K and M are square matrices and are referred to herein as the stiffness matrix and the mass matrix respectively. The vector ϕ defines the natural mode. When K and M are symmetric, and K or M is positive definite, the eigenvalues, ω^2, are real; when both K and M are positive definite, ω^2 is always positive [Ref. 1]. When K and M are developed using energy techniques, they are always symmetric; however, when they are developed for the same problem by applying finite differences to the governing partial differential equations of motion and the boundary conditions, they frequently emerge numerically as unsymmetric, calling into question whether numerical results for ω^2 will in fact be real.

In this thesis an unsymmetric stiffness matrix is developed by finite differences for the cantilever beam. A perturbation theory for real symmetric matrices given

by Wilkinson [Ref. 2] is used to establish that all of the eigenvalues of the unsymmetric matrix formulation are real and positive; hence the results obtained for the natural frequencies are real, representing undamped harmonic motion as assumed.

II. DERIVATION OF THE MATRIX EQUATIONS

The structure selected for analysis is the cantilever beam of length L shown in Figure 1.



BEAM CONFIGURATION
FIGURE I

The equation governing the free undamped vibration of the beam can be given in the form

$$\frac{\partial^4 \Phi}{\partial x^4} = - \mu \frac{\partial^2 \Phi}{\partial t^2} \qquad (2\text{-}1)$$

in which t is time, and

$$\Phi = Y/L$$

$$x = X/L$$

$$\mu = \rho/(Er^2)$$

where Y is the deflection of the beam, X is the axial coordinate, ρ is the mass density, E is the elastic modulus, and r is the radius of gyration of the cross-section.

Assuming harmonic motion,

$$\Phi = \phi \sin \omega t \qquad (2\text{-}2)$$

Substituting Equation 2-2 into Equation 2-1 leads to

$$\frac{d^4\phi}{d x^4} = \lambda\phi \tag{2-3}$$

in which $\lambda = \mu\omega^2$.

The boundary conditions on the beam are

$$\phi = \frac{d\phi}{dx} = 0 \qquad \text{at } x = 0 \tag{2-4a}$$

and

$$\frac{d^2\phi}{dx^2} = \frac{d^3\phi}{dx^3} = 0 \qquad \text{at } x = 1 \tag{2-4b}$$

Let the beam be divided into ten equal intervals of length hL as shown in Figure 1. The nodes j=-1, j=11, and j=12 are fictitious nodes located off of the beam. Let the derivatives in Equation 2-3 and Equations 2-4 be approximated by

$$\left(\frac{d\phi}{dx}\right)_j \doteq (-\phi_{j-1} + \phi_{j+1}) / (2h) \tag{2-5a}$$

$$\left(\frac{d^2\phi}{dx^2}\right)_j \doteq (\phi_{j-1} - 2\phi_j + \phi_{j+1}) / h^2 \tag{2-5b}$$

$$\left(\frac{d^3\phi}{dx^3}\right)_j \doteq (-\phi_{j-2} + \phi_{j-1} - \phi_{j+1} + \phi_{j+2}) / (2h^3) \tag{2-5c}$$

$$\left(\frac{d^4\phi}{dx^4}\right)_j \doteq (\phi_{j-2} - 4\phi_{j-1} + 6\phi_j - 4\phi_{j+1} + \phi_{j+2})/h^4 \tag{2-5d}$$

in which h=0.1. The truncation error of each of the finite difference approximations in Equations 2-5 is $O(h^2)$. Applying Equation 2-3 at j=1,2,3, ... 10, Equations

2-4a at j=0, and Equations 2-4b at j=10; and eliminating the variables ϕ_{-1}, ϕ_0, ϕ_{11}, and ϕ_{12} eventually leads to the matrix equation

$$\begin{bmatrix} 7 & -4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdot & 0 \\ 0 & \cdot & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & \cdot & \cdot & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 2 & -4 & 2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \end{Bmatrix} = \lambda h^4 I \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \\ \phi_{10} \end{Bmatrix} \qquad (2\text{-}6)$$

where I is the identity matrix. Note that the stiffness matrix in Equation 2-6 is not symmetric. However, if the last row of Equation 2-6 is divided by two, the resulting formulation is symmetric. This results in the mass matrix M no longer being equal to I, since the diagonal element in the last row is equal to one-half; however, M is still symmetric and positive definite, and all eigenvalues are real. The symmetric stiffness matrix will be referred to hereafter as K_1.[1] Note that all of the finite difference equations used to construct K_1 are centered about j.

---

[1]It can be shown that this symmetric formulation is identical to a matrix equation developed using energy techniques with one-half of a mass interval at j=10.

An alternate matrix formulation of the governing difference equations can be constructed in a similar manner by using the $O(h^2)$ approximation [Ref. 3]

$$\left(\frac{d^3\phi}{dx^3}\right)_j \doteq (\phi_{j-3}-6\phi_{j-2}+12\phi_{j-1}-10\phi_j+3\phi_{j+1})/(2h^3) \tag{2-7}$$

instead of Equation 2.5c.

As a consequence of the fact that Equation 2-7 is not centered about j, the fictitious node j=12 is not required, and Equation 2-3 is applied at j=1,2,3,........9. The boundary conditions are satisfied by applying Equations 2-4a at j=0 and Equations 2-4b at j=10. Eliminating ϕ_{-1}, ϕ_0, ϕ_{10}, and ϕ_{11} leads to the unsymmetric matrix formulation

$$\begin{bmatrix} 7 & -4 & 1 & 0 & . & . & . & . & 0 \\ -4 & 6 & -4 & 1 & 0 & . & . & . & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & . & . & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & . & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & . & . & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & . & . & . & 0 & 1 & -3\frac{3}{4} & -4\frac{1}{2} & -1\frac{3}{4} \\ 0 & . & . & . & . & 0 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{Bmatrix} = \lambda h^4 I \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \\ \phi_9 \end{Bmatrix} \tag{2-8}$$

The stiffness matrix in Equation 2-8 will be referred to as K_2.

III. ANALYSIS AND RESULTS

A perturbation theory given by Wilkinson [Ref. 2] for real symmetric matrices will be used to prove that the eigenvalues of the real unsymmetric formulation given by Equation 2-8 are real. The brief description of the theory which follows is expanded in the Appendix.

Let

$$K_2 = A + B$$

where K_2 is the real unsymmetric square matrix of order n, A is an n x n real symmetric matrix, and B is an n x n real unsymmetric matrix. Every eigenvalue of K_2 lies within at least one of the n circular discs whose centers are the eigenvalues of A. The radius of these circular discs is $\|B\|_2$, the 2-norm of B, where

$$\|B\|_2 = (\text{maximum eigenvalue of } B^H B)^{\frac{1}{2}}$$

and B^H is the Hermitian conjugate matrix of B. The matrix $B^H B$ is then Hermitian, the eigenvalues of which are all real. If any m of the n circular discs form a connected domain isolated from the others, there are precisely m eigenvalues in this connected domain. In the case of real unsymmetric matrices, any complex eigenvalues must occur as conjugate pairs. If the i^{th} circular disc is isolated and contains only one eigenvalue, this eigenvalue must therefore be real.

The matrix K_2 can be decomposed in many ways to obtain the matrices A and B. One technique is to use the symmetric

matrix K_1 as A and obtain B by subtraction. This method was tried and resulted in such a large $\|B\|_2$ that only two eigenvalues of K_2 could be proven real. Consequently, two other approaches were considered in an attempt to develop a smaller norm size that would prove all the eigenvalues to be real and positive. These two approaches are referred to as Method I and Method II.

In Method I, the matrix K_2 is decomposed such that B contains the smallest <u>number</u> of elements possible, i.e.,

$$
\underbrace{\begin{bmatrix} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -4 & 6 & -4 & 2 & 0 \\ \cdots & 2 & -4 & 6 & -4 & 1 \\ \cdots & 0 & 1 & -3 & 4\frac{1}{2} & -1 \\ \cdots & \cdots & 0 & \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}}_{K_2} =
$$

$$
\underbrace{\begin{bmatrix} & \cdots & \cdots & \cdots & \cdots & \cdots \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 4\frac{1}{2} & -1 \\ \cdots & 0 & 1 & -1 & \frac{1}{2} \end{bmatrix}}_{A_I} + \underbrace{\begin{bmatrix} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & -\frac{1}{4} & 0 & -\frac{3}{4} \\ \cdots & 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}}_{B_I}
$$

Equation 3-1 shows only that portion of K_2 that requires alteration. The eigenvalues of A_I and the 2-norm of B_I were determined using the IBM SSP subroutine EIGENP and are tabulated in Table I and plotted in Figure 2. Note in Figure 2 that five of the circular discs are isolated and in the positive domain. As a result, applying the perturbation theory proves that at least five of the nine

eigenvalues are real and positive. Also note that the five real eigenvalues are the five largest eigenvalues. In order to prove that the lower eigenvalues in the overlapping circular discs are real, the inverse formulation

$$\frac{1}{\lambda} I \phi = K_2^{-1} \phi \qquad (3\text{-}2)$$

is examined. If $1/\lambda$ can be proven real and positive, then λ is real and positive.

TABLE I

EIGENVALUES AND NORMS - METHOD I

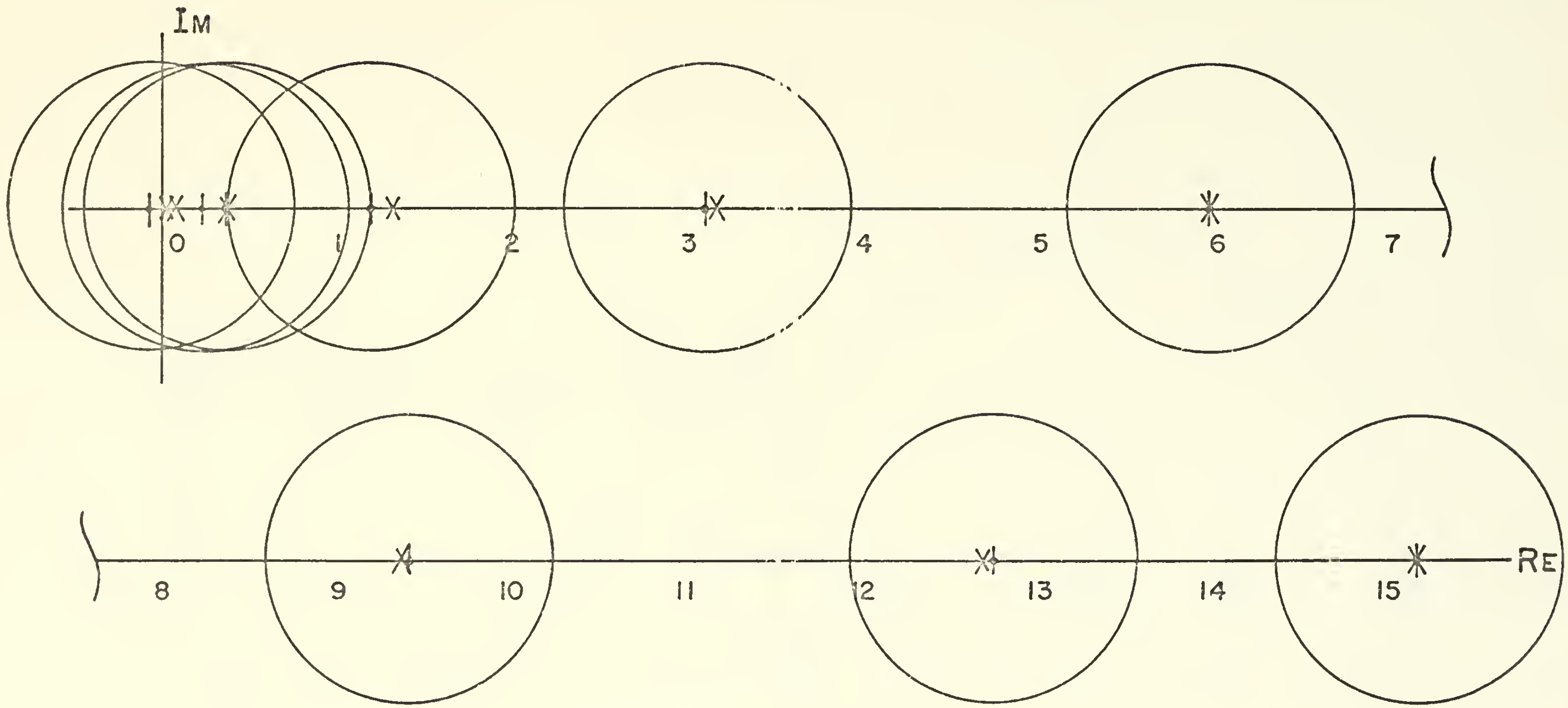
i	Eigenvalue λ:			
	A_I	K_2	A_I'	K_2^{-1}
1	-0.079	0.0013	0.067	0.066
2	0.222	0.049	0.083	0.079
3	0.361	0.369	0.119	0.107
4	1.19	1.31	0.209	0.168
5	3.10	3.16	0.472	0.316
6	5.97	5.96	1.57	0.764
7	9.40	9.36	10.63	2.71
8	12.72	12.68	91.41	20.42
9	15.12	15.11	720.2	800.1

Value of $\|B_I\|_2 = 0.814$

Value of $\|B_{I'}\|_2 = 180.9$

FIGURE 2

METHOD I RESULTS IN THE COMPLEX-PLANE



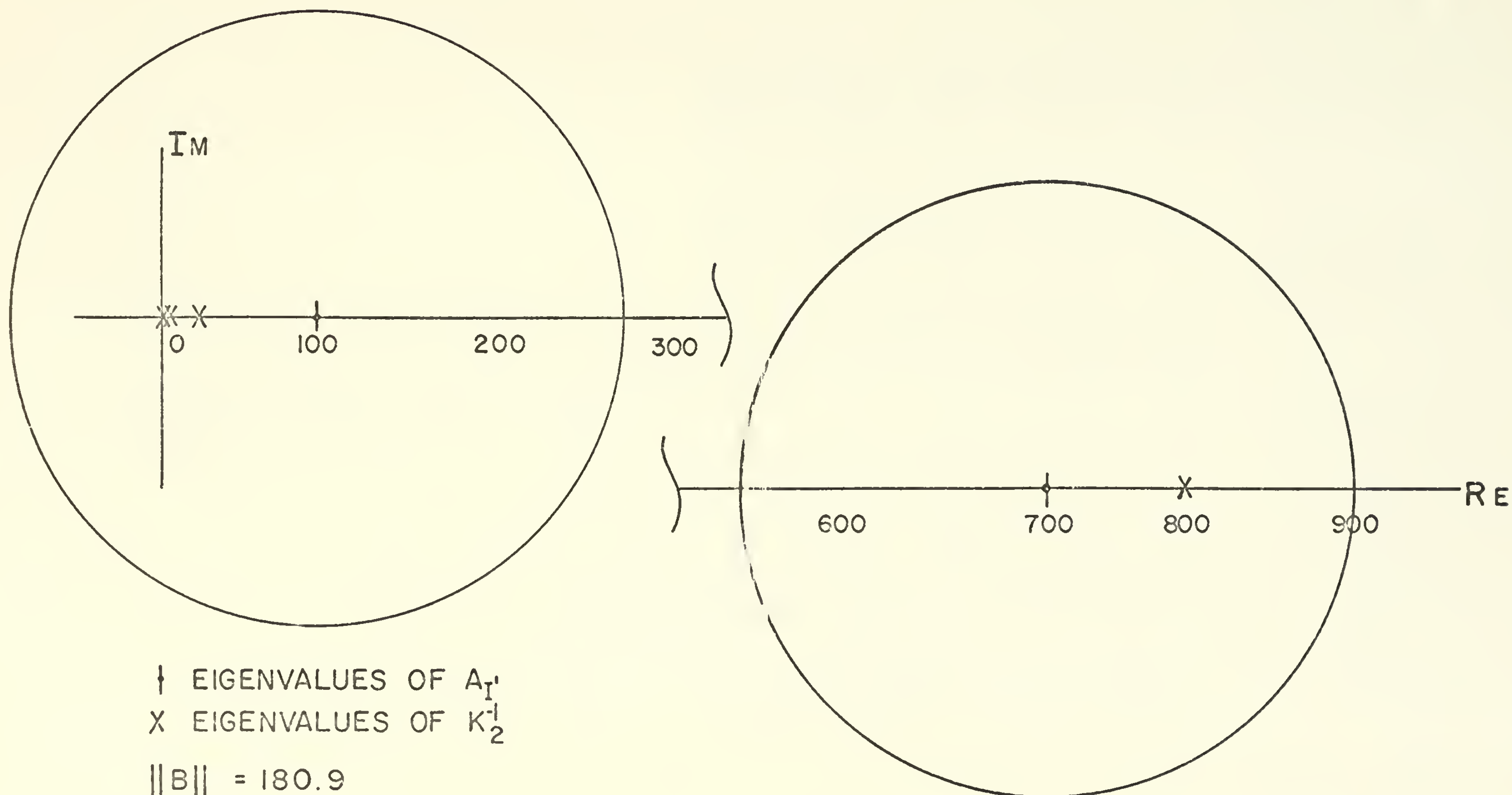
⍿ EIGENVALUES OF A_I

X EIGENVALUES OF K_2

$\|B\|_2 = 0.814$

FIGURE 3

METHOD I RESULTS IN THE COMPLEX-PLANE



ǀ EIGENVALUES OF A_I'

X EIGENVALUES OF K_2^{-1}

$||B||_2 = 180.9$

Also, inversion directly relates the lowest eigenvalue of K_2 to the largest eigenvalue of K_2^{-1}, and so on. Since the lower eigenvalues of K_2 have not yet been proven real, the higher eigenvalues of K_2^{-1} are of particular interest.

The decomposition of K_2^{-1} for Method I is

$$\begin{bmatrix} \frac{1}{2} & 1 & 1\frac{1}{2} & 2 & 2\frac{1}{2} & 3 & 3\frac{1}{2} & 4 & 7 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 26\frac{1}{2} \\ 1\frac{1}{2} & 5 & 9\frac{1}{2} & 14 & 18\frac{1}{2} & 23 & 27\frac{1}{2} & 32 & 57 \\ 2 & 7 & 14 & 22 & 30 & 38 & 46 & 54 & 97 \\ 2\frac{1}{2} & 9 & 18\frac{1}{2} & 30 & 42\frac{1}{2} & 55 & 67\frac{1}{2} & 80 & 145 \\ 3 & 11 & 23 & 38 & 55 & 73 & 91 & 109 & 199\frac{1}{2} \\ 3\frac{1}{2} & 13 & 27\frac{1}{2} & 46 & 67\frac{1}{2} & 91 & 115\frac{1}{2} & 140 & 259 \\ 4 & 15 & 32 & 54 & 80 & 109 & 140 & 172 & 322 \\ 4\frac{1}{2} & 17 & 36\frac{1}{2} & 62 & 92\frac{1}{2} & 127 & 164\frac{1}{2} & 204 & 387 \end{bmatrix} =$$

K_2^{-2}

$$\begin{bmatrix} \frac{1}{2} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & 4\frac{1}{2} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 15 & 17 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 32 & 36\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 54 & 62 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 80 & 92\frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 109 & 127 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 140 & 164\frac{1}{2} \\ 4 & 15 & 32 & 54 & 80 & 109 & 140 & 172 & 204 \\ 4\frac{1}{2} & 17 & 36\frac{1}{2} & 62 & 92\frac{1}{2} & 127 & 164\frac{1}{2} & 204 & 387 \end{bmatrix} +$$

$A_I{}'$

$$
\begin{bmatrix}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 2\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 9\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 20\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 35 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 52\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 73\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 94\frac{1}{2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 118 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0
\end{bmatrix} \qquad (3\text{-}3)
$$

$B_{I'}$

The eigenvalues of $A_{I'}$ and the 2-norm of $B_{I'}$ are also shown in Table I and in Figure 3. Examination of Figure 3 reveals that the largest eigenvalue of K_2^{-1} is real and positive; hence, the lowest eigenvalue of K_2 is real and positive. Thus, six of the nine eigenvalues have been proven real and positive. To verify that these eigenvalues are in fact real and positive, the actual eigenvalues of K_2 and K_2^{-1} were computed using EIGENP and are tabulated in Table I and plotted in Figure 2 and Figure 3. They do in fact lie within the circular discs.

The decomposition of K_2 and K_2^{-1} of Method I led to a relatively large norm and the proof that only six of the nine eigenvalues were real and positive. Consider now Method II in which K_2 is decomposed such that the <u>moduli</u> of the elements of B are as small as possible (making

B skew-symmetric), i.e.,

$$\left.\begin{array}{ccccc} & & & & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & -4 & 1 & 0 \\ \cdot & -4 & 6 & -4 & 1 \\ \cdot & 2 & -3\frac{3}{4} & 4\frac{1}{2} & -1\frac{3}{4} \\ \cdot & 0 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] = \qquad (3\text{-}4)$$

$$K_2$$

$$\left.\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & -4 & 1 & 0 \\ \cdot & -4 & 6 & -3\frac{7}{8} & \frac{3}{4} \\ \cdot & 1 & -3\frac{7}{8} & 4\frac{1}{2} & -1\frac{3}{8} \\ \cdot & 0 & \frac{3}{4} & -1\frac{3}{8} & \frac{1}{2} \end{array}\right] + \left.\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & -\frac{1}{8} & \frac{1}{4} \\ \cdot & 0 & \frac{1}{8} & 0 & -\frac{3}{8} \\ \cdot & 0 & -\frac{1}{4} & \frac{3}{8} & 0 \end{array}\right]$$

$$A_{II} \qquad\qquad\qquad\qquad B_{II}$$

Only that portion of K_2 requiring alteration is shown in Equation 3-4. Using the same procedures as in Method I leads to the results tabulated in Table II and plotted in Figure 4. The norm of B_{II} is much smaller than the norm of B_I, and examination of Figure 4 reveals that seven of the nine eigenvalues of K_2 are real and positive. Decomposing K_2^{-1} in a manner similar to Equation 3-4 leads to A_{II}' and B_{II}'. The results are also shown in Table II and in Figure 5. Using this formulation, the lowest eigenvalue of K_2 is proven real and positive. Consequently, use of Method II proves eight of the nine eigenvalues to be real

and positive. The last eigenvalue must also be real since it can have no conjugate. It must also be positive since it is larger than the lowest eigenvalue which has been proven positive. Therefore all the eigenvalues of K_2 are real and positive.

A third method considered is to treat K_2 in a manner similar to the conversion of Equation 2-6 to the symmetric formulation K_1. In this method, K_2 is altered by multiplying row 9 by two. Although this leads to another unsymmetric matrix, a much smaller $\|B\|_2$ can be developed. However, from past experience it is concluded that this approach would not prove all eigenvalues real and positive without resorting to the inversion of the matrix. Since Method II proved the eigenvalues of K_2 real and positive without alteration of the matrix, and since the eigenvalues of this third matrix formulation are not $\mu\omega^2$, this approach is not pursued further.

As a check on the accuracy of the calculated eigenvalues of K_2, a comparison of the exact natural frequencies [Ref. 4] and the numerical natural frequencies is made in Table III for the five lowest frequencies.

TABLE II

EIGENVALUES AND NORMS - METHOD II

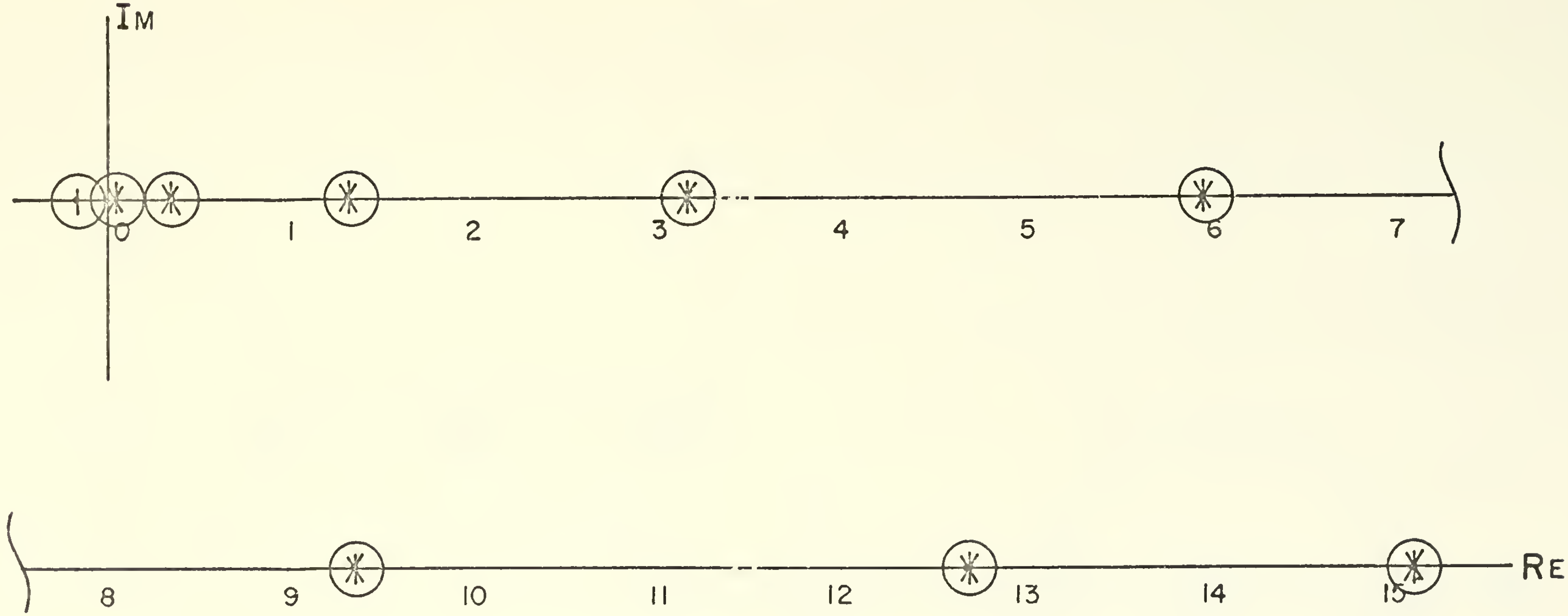
i	Eigenvalue λ:			
	A_{II}	K_2	A_{II}'	K_2^{-1}
1	-0.144	0.0013	-5.58	0.066
2	0.040	0.049	0.070	0.079
3	0.363	0.369	0.084	0.107
4	1.31	1.31	0.119	0.168
5	3.17	3.16	0.209	0.316
6	5.97	5.96	0.494	0.764
7	9.37	9.36	1.72	2.71
8	12.68	12.68	17.10	20.42
9	15.11	15.11	810.6	800.1

Value of $\|B_{II}\|_2 = 0.148$

Value of $\|B_{II}'\|_2 = 90.3$

FIGURE 4

METHOD II RESULTS IN THE COMPLEX-PLANE



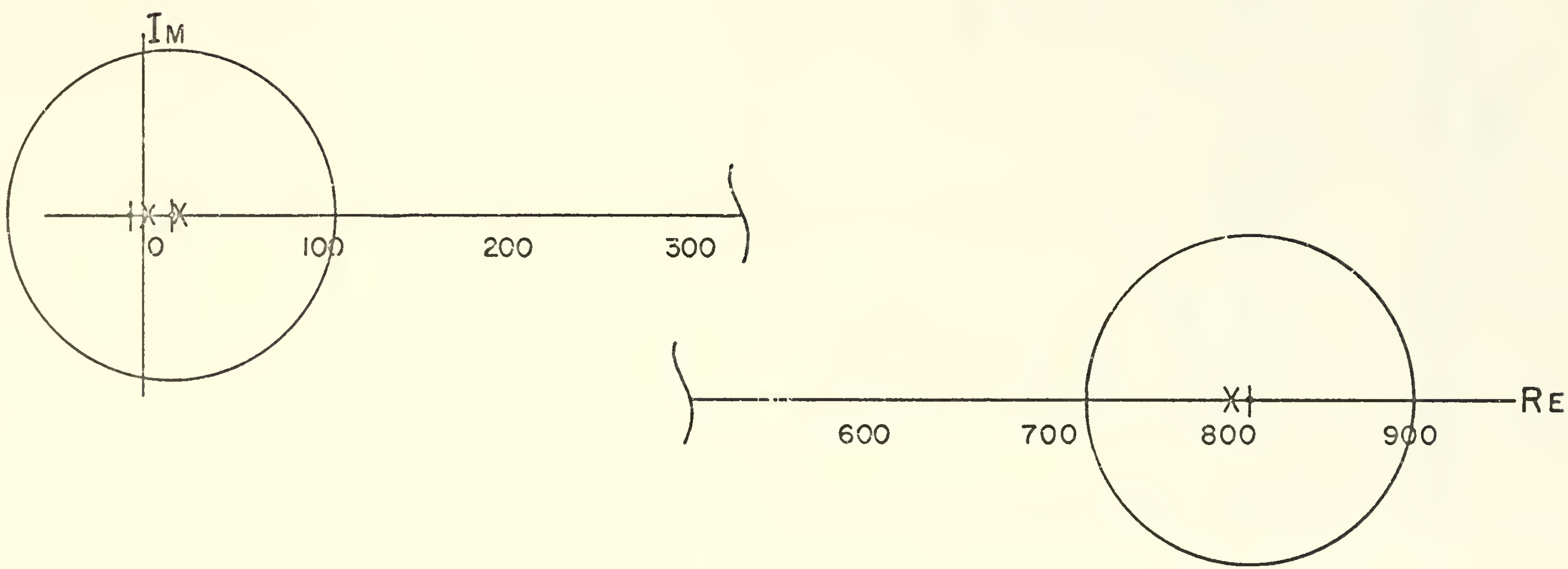
| EIGENVALUES OF A_{II}

X EIGENVALUES OF K_2

$\|B\|_2 = 0.148$

FIGURE 5

METHOD II RESULTS IN THE COMPLEX-PLANE



| EIGENVALUES OF A_{II}'

X EIGENVALUES OF K_2^{-1}

$\|B\|_2 = 90.3$

TABLE III

NATURAL FREQUENCIES OF CANTILEVER BEAM

Natural Frequency rad/sec	
Exact [Ref. 4]	Numerical
0.560	0.562
3.510	3.522
9.820	9.671
19.20	18.210
31.80	28.300

IV. MULTIPLE VARIABLE FORMULATION

An alternate unsymmetric formulation is obtained when the two variables ϕ and $\mathcal{M}$, where $\mathcal{M}$ is the bending moment in the beam, are used instead of only ϕ. The governing equations are

$$\frac{\partial^2 \phi}{\partial x^2} = \mathcal{M} \tag{4-1}$$

and

$$\frac{\partial^2 \mathcal{M}}{\partial x^2} = \lambda \phi \tag{4-2}$$

The boundary conditions for the two-variable formulation are

$$\phi = \frac{d\phi}{dx} = 0 \qquad \text{at } x=0 \tag{4-3}$$

and

$$\mathcal{M} = \frac{d\mathcal{M}}{dx} = 0 \qquad \text{at } x=1 \tag{4-4}$$

Consider the beam shown in Figure 1. Applying Equation 4-1 and Equation 4-2 at j=1,2,3,.........10, Equation 4-3 at j=0, Equation 4-4 at j=10, and eliminating ϕ_{-1}, ϕ_o, ϕ_{11}, $\mathcal{M}_{-1}$, $\mathcal{M}_o$, and $\mathcal{M}_{11}$ leads to the matrix equation

$$(4\text{-}5)$$

$$
\begin{bmatrix}
\frac{2}{h^2} & -2 & 0 & 1 & 0 & \cdot & & & & \\
-2 & -h^2 & 1 & 0 & 0 & \cdots & & & & \\
0 & 1 & 0 & -2 & 0 & 1 & 0 & \cdot\cdot & & \\
1 & 0 & -2 & -h^2 & 1 & 0 & 0 & \cdot\cdot & & \\
0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & \cdot\cdot \\
0 & 0 & 1 & 0 & -2 & -h^2 & 1 & 0 & 0 & \cdot\cdot \\
 & & & & & & & & & \\
\cdot & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\
\cdot & 0 & 1 & 0 & -2 & -h^2 & 1 & 0 & 0 & 0 \\
 & & \cdot\cdot & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\
 & & \cdot\cdot & 0 & 1 & 0 & -2 & -h^2 & 1 & 0 \\
 & & & & \cdot\cdot & 0 & 0 & 2 & 0 & 0 \\
 & & & & \cdot & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix}
\phi_1 \\ m_1 \\ \phi_2 \\ m_2 \\ \phi_3 \\ m_3 \\ \vdots \\ \phi_8 \\ m_8 \\ \phi_9 \\ m_9 \\ \phi_{10} \\ m_{10}
\end{Bmatrix}
=
$$

$$
\lambda h^2
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdot\cdot & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdot\cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & & & & & & & & & \\
\cdot\cdot & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\cdot\cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 1 & 0 \\
 & & & & \cdots & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\phi_1 \\ m_1 \\ \phi_2 \\ m_2 \\ \phi_3 \\ m_3 \\ \vdots \\ \phi_8 \\ m_8 \\ \phi_9 \\ m_9 \\ \phi_{10} \\ m_{10}
\end{Bmatrix}
$$

Equation 4-5 is unsymmetric. However, if the variables η_1, η_2, ... η_{10} are eliminated, Equation 4-5 reduces to Equation 2-6. As previously discussed, Equation 2-6 is simply a symmetric matrix in disguise, so that the eigenvalues are real and positive.

The two-variable formulation has been presented here to point out that many unsymmetric matrices may in fact be reducible to symmetric matrices; hence, they have all of the properties of symmetric matrices.

V. CONCLUSIONS AND RECOMMENDATIONS

The basic concept illustrated in this thesis is that for a given unsymmetric matrix the eigenvalues may be examined for proof that they are real and positive without actually calculating them. Use of the perturbation theory given by Wilkinson gives a relatively straightforward method by which this may be accomplished. For the simple example selected calculation of the eigenvalues is easily carried out, thus confirming the results.

It is recommended that this technique be applied to other eigenvalue problem formulations of more involved physical systems. If the technique proves itself reliable in application to matrices of large order, significant computer time can be saved by ascertaining whether a given modelling method will result in real, positive eigenvalues prior to conducting lengthy eigenvalue computations.

APPENDIX

PERTURBATION THEORY GIVEN BY WILKINSON

With slight changes of notation and text, the following material is reproduced for ease of reference from the pages cited for Reference 2.

Consider an eigenvalue λ (possibly complex) of the real system

$$(A + B)x = \lambda x \qquad \text{(A-1a)}$$

in which A is symmetric but B is not. Since A is real there is an orthogonal matrix H such that

$$H^{-1}AH = \text{diag}\ (\lambda_i) \qquad \text{(A-1b)}$$

and the λ_i are real. Because λ is an eigenvalue of (A+B), the matrix $(A+B-\lambda I)$ is singular and hence its determinant is zero. From Equation A-1a and Equation A-1b

$$H^{-1}(A+B-\lambda I)H = \text{diag}\ (\lambda_i - \lambda) + H^{-1}BH \qquad \text{(A-2)}$$

and taking determinants, the matrix on the right of Equation A-2 must also be singular. Two cases are distinguished.

Case 1. $\lambda = \lambda_i$ for some i

Case 2. $\lambda \neq \lambda_i$ for any i, so that

$$\text{diag}(\lambda_i - \lambda) + H^{-1}BH = \text{diag}(\lambda_i - \lambda)\left[I + \text{diag}(\lambda_i - \lambda)^{-1} H^{-1}BH \right] \qquad \text{(A-3)}$$

and taking determinants again, the matrix in brackets must be singular. If (I + Z), where Z is a general matrix, is singular, $\|Z\|_2 \geq 1$; for if $\|Z\|_2 < 1$, none of the

eigenvalues $(I + Z)$ can be zero [Ref. 5]. Hence

$$\left\| \operatorname{diag} (\lambda_i - \lambda)^{-1} H^{-1} B H \right\|_2 \doteq 1 \qquad \text{(A-4)}$$

giving

$$\max \left| \lambda_i - \lambda \right|^{-1} \left\| H^{-1} \right\|_2 \left\| B \right\|_2 \left\| H \right\|_2 \geq 1 \qquad \text{(A-5)}$$

i.e. (in both Case 1 and Case 2)

$$\min \left| \lambda_i - \lambda \right| \leq \left\| H^{-1} \right\|_2 \left\| H \right\|_2 \left\| B \right\|_2 \qquad \text{(A-6)}$$

for at least one λ_i for every eigenvalue λ. That is to say, every eigenvalue of $(A + B)$ lies in at least one of the circular discs given by

$$\left| \lambda_i - \lambda \right| \leq \left\| H^{-1} \right\|_2 \left\| H \right\|_2 \left\| B \right\|_2 \qquad \text{(A-7)}$$

and since H is orthogonal (all eigenvalues of which have modulus one), these reduce to

$$\left| \lambda_i - \lambda \right| \leq \left\| B \right\|_2 \qquad \text{(A-8)}$$

Further, if any s of these discs form a connected domain isolated from the others, then there are precisely s eigenvalues in this domain.

In the case of a real unsymmetric matrix such as $(A + B)$, any complex eigenvalues must occur in conjugate pairs. If $\left| \lambda_i - \lambda_j \right| > 2 \left\| B \right\|_2$ $(j \neq i)$, then the i^{th} circular disc is isolated and contains only one eigenvalue. This eigenvalue must therefore be real. Hence, if all eigenvalues of A are separated by more than $2 \left\| B \right\|_2$, all eigenvalues of $(A + B)$ are real.

LIST OF REFERENCES

1. Crandall, S. H., Engineering Analysis, pp. 75-76, McGraw-Hill, 1956.

2. Wilkinson, J. H., The Algebraic Eigenvalue Problem, pp. 55-56, 71, 87-88, 93-94, Clarendon Press, 1965.

3. Collatz, L., The Numerical Treatment of Differential Equations, p. 539, Springer-Verlag, 1960.

4. Rogers, G. L., Dynamics of Framed Structures, p. 176, Wiley, 1959.

5. Bauer, F. L. and Fike, C. T., Norms and Exclusion Theorems, pp. 137-141, Numerische Mathematic, v. 2, 1960.

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13. ABSTRACT

This thesis is concerned with proving that the eigenvalues of a specific unsymmetric matrix are real and positive, without actually computing them. The method of finite differences is applied to the vibration analysis of a cantilever beam and leads to an unsymmetric stiffness matrix in the eigenvalue problem formulation. The technique employed in the proof is based on a perturbation theory given by Wilkinson for real symmetric matrices. Application of the theory is made to the cantilever beam eigenvalue problem. The results verify that the eigenvalues of this and other unsymmetric matrices can be proven real and positive without their actual values being calculated.

A multiple-variable formulation of the cantilever beam vibration analysis is also examined to illustrate that it implicitly contains all of the properties of a symmetric matrix.

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